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The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity*

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Abstract

In this paper we present a very simple proof of the existence of at least one non trivial solution for a Kirchhoff type equation on $\mathbb{R}^N$, for $N \geqslant 3$. In particular, in the first part of the paper we are interested in studying the existence of a positive solution to the elliptic Kirchhoff equation under the effect of a nonlinearity satisfying the general Berestycki-Lions assumptions. In the second part we look for ground states using minimizing arguments on a suitable natural constraint.

keywords: Kirchhoff equation, Pohozaev identity, natural constraint, minimizing sequence
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Introduction

The multidimensional Kirchhoff equation is

$$\frac{\partial^2 u}{(\partial t)^2} - (1 + \int_\Omega |\nabla u|^2)\Delta u = 0 \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ satisfies some initial or boundary conditions. It arises from the following Kirchhoff' nonlinear generalization (see [5]) of the well known d'Alembert equation

$$\rho \frac{\partial^2 u}{(\partial t)^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{(\partial x)^2} = 0. \tag{2}$$

Equation (2) describes a vibrating string, taking into account the changes in length of the string during the vibration. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. In [6] the problem was proposed in the following form:

$$\begin{cases} \frac{\partial^2 u}{(\partial t)^2} - M(\int_\Omega |\nabla u|^2)\Delta u = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{in } \partial\Omega \times (0, T) \\ u(0) = u_0, \quad u'(0) = u_1, & \end{cases}$$

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where $M : [0,+\infty[\to \mathbb{R}$ is a continuous function such that $M(s) \geqslant c > 0$ for any $s \geqslant 0$, and Ω is a bounded set in $\mathbb{R}^N$, with smooth boundary.
This hyperbolic problem has an elliptic version when we look for static solutions.
In [14], it has been considered a class of problems among which the following elliptic Kirchhoff type equation was included

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = f & \text{in } \Omega,\\ u = 0 & \text{in } \partial\Omega \end{cases}$$

(here Ω is an open subset of $\mathbb{R}^N$).
Taking into account the original formulation of the equation given by Kirchhoff, we assume the following

Definition 0.1. *If there exist two positive constants a and b such that $M : \mathbb{R}_+ \to \mathbb{R}$ can be written $M(s) = a + bs$, then M is called Kirchhoff function.*

Recently, many authors have used variational methods to study the Kirchhoff equation perturbed by a local nonlinear term (see [7] for a short survey on the topic). By arguments based on the mountain pass theorem, in [1] the problem

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{in } \partial\Omega \end{cases}$$

has been solved in a bounded domain of $\mathbb{R}^N$ under suitable growth conditions on $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $M : [0,+\infty[\to \mathbb{R}$. Taking $N = 1, 2$ or 3, the problem has been treated also in [10] where M is a Kirchhoff function, and the nonlinearity $f(x,t)$ has been supposed to behave linearly near 0 and like t^3 at infinity. The Yang index has been used to find a nontrivial solution. A similar growth at infinity has been assumed in [15] where the authors have looked for sign changing solutions. They also have obtained a sign changing solution when the nonlinearity f satisfies either the following growth condition

$$|f(x,t)| \leqslant C(|t|^{p-1} + 1), \text{ uniformly in } x, \text{ for } p < 4$$

or the following Ambrosetti-Rabinowitz condition

$$\nu F(x,t) \leqslant t f(x,t), \text{ for } |t| \text{ large and } \nu > 4.$$

In [8], the equation has been studied assuming that the nonlinearity grows at infinity more than t^3, without introducing the Ambrosetti-Rabinowitz hypothesis. Using a variational approach, a multiplicity result has been showed in [4].
Finally we recall the recent result obtained in [11], where three solutions have been found for the Kirchhoff type problem

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = \lambda f(x,u) + \mu g(x,t) & \text{in } \Omega,\\ u = 0 & \text{in } \partial\Omega \end{cases}$$

where λ and μ are two parameters.

The joining point of all these papers is that they consider the equation on a bounded domain, with Dirichlet conditions on the boundary.
In this paper we study an autonomous Kirchhoff type equation on all the space

$\mathbb{R}^N$, looking for the existence of positive solutions, namely we consider the problem

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = g(u) & \text{in } \mathbb{R}^N,\ N \geqslant 3, \\ u > 0. \end{cases} \tag{$\mathcal{K}$}$$

In this sense, the problem turns out to be a generalization of the well known Schrödinger equation

$$-\Delta u = g(u), \quad \text{in } \mathbb{R}^N. \tag{$\mathcal{S}$}$$

In the first part of the paper we are interested in studying the problem ($\mathcal{K}$) in presence of a Berestycki-Lions nonlinearity. In order to explain what this means, we provide the following definitions

Definition 0.2. *A function* $g : \mathbb{R} \to \mathbb{R}$ *is called a Berestycki-Lions nonlinearity if it satisfies the following assumptions*

(**g1**) $g \in C(\mathbb{R}, \mathbb{R})$, $g(0) = 0$;

(**g2**) $-\infty < \liminf_{s\to 0^+} g(s)/s \leqslant \limsup_{s\to 0^+} g(s)/s = -m < 0$;

(**g3**) $-\infty \leqslant \limsup_{s\to +\infty} g(s)/s^{2^*-1} \leqslant 0$;

(**g4**) *there exists* $\zeta > 0$ *such that* $G(\zeta) := \int_0^\zeta g(s)\, ds > 0$.

A function $g : \mathbb{R} \to \mathbb{R}$ *is called a zero mass Berestycki-Lions nonlinearity if it satisfies* (**g1**), (**g3**), (**g4**) *and the following zero mass assumption*
(**g2**)′ $-\infty < \liminf_{s\to 0^+} g(s)/s, \quad \limsup_{s\to 0^+} g(s)/s^{2^*-1} \leqslant 0$.

Remark 0.3. *Using the terminology inherited by [2], we refer to the constant* m *in* (**g2**) *calling it mass. This is the reason for which we say that a function* g *satisfying* (**g2**)′ *instead of* (**g2**) *is a zero mass nonlinearity.*

In the very celebrated paper [2], these types of nonlinearities appeared for the first time, and it was showed that the hypotheses (**g1**),. . .,(**g4**) are *almost* optimal to get an existence result for the Schrödinger equation.

In the first section of this paper, we use a simple rescaling argument to establish a sufficient condition for the existence of a solution to ($\mathcal{K}$). The first result we get is the following

Theorem 0.4. *Let* $M : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ *a continuous function such that*

$$\liminf_{t\to 0} tM(t^{\frac{2-N}{2}}) = 0 \tag{3}$$

and let $g : \mathbb{R} \to \mathbb{R}$ *a (possibly zero mass) Berestycki-Lions nonlinearity. Then the problem* ($\mathcal{K}$) *has a solution in* $C^2(\mathbb{R}^N)$.

We point out that any Kirchhoff function satisfies our assumptions if $N = 3$. On the other hand, when M is a Kirchhoff type function we are able to refine our estimates and we get the following result

Theorem 0.5. *Let* $g : \mathbb{R} \to \mathbb{R}$ *a (possibly zero mass) Berestycki-Lions nonlinearity,* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *a continuous function and* $M(s) = a + bf(s)$. *Then for any* $N \geqslant 3$ *there exists a positive constant* δ *(depending on* a*) such that if* $b \in]0, \delta]$, *the problem* ($\mathcal{K}$) *has a solution in* $C^2(\mathbb{R}^N)$. *If moreover*

$$\liminf_{t\to 0} t^{\frac{2}{2-N}} f(t) = 0, \tag{4}$$

then there exists a positive constant δ (depending on b) such that if $a \in]0, \delta]$, the problem $(\mathcal{K})$ has a solution in $C^2(\mathbb{R}^N)$.

We again remark that when M is a Kirchhoff function, then $f = id|_{\mathbb{R}_+}$ and assumption (4) is automatically satisfied for $N \geqslant 5$.

In the second part of the paper we study the existence of the so called *ground state* solutions to the Kirchhoff equation

$$-(a + b\int_{\mathbb{R}^N} |\nabla u|^2)\Delta u = g(u). \tag{5}$$

We recall that a ground state is a solution which minimizes the functional of the action among all the other solutions.

The problem of finding such a type of solutions is a very classical problem: it has been introduced by Coleman, Glazer and Martin in [3], and reconsidered by Berestycki and Lions in [2] for a class of nonlinear equations including the Schrödinger's one. Here we will use a minimizing argument based on an idea developed in [13] (recent applications can be found in [9] and in [12]). In that paper the author showed that the ground state for the Schrödinger equation can be found as the minimizer of the functional of the action restricted to a particular natural constraint. This natural constraint actually is a manifold which is constituted by all non null functions satisfying the Pohozaev identity related to the Schrödinger equation.
Unfortunately, a similar manifold turns out to be a nice constraint in order to look for ground state solutions of the Kirchhoff equation only if $N = 3$ or $N = 4$: when $N \geqslant 5,$ we are no more able to establish if the restricted functional is bounded below. The final result we get is

Theorem 0.6. *If $N = 3$ or $N = 4$ and g is a Berestycki-Lions nonlinearity, then equation* (5) *possesses a ground state solution.*

The paper is so organized:

in Section 1 we show our rescaling argument to get a solution for $(\mathcal{K})$ in the general case described in Theorem 0.4 and in the particular situation of a Kirchhoff type function as in Theorem 0.5.

In Section 2 we study the problem of the existence of a ground state solution for the Kirchhoff equation using a variational approach.

1 Bound state solution

In the sequel, we denote by v a ground state solution of $(\mathcal{S})$ (respectively a bound state solution if g is a zero mass Berestycki-Lions nonlinearity).

Proof of Theorem 0.4 Observe that, by hypothesis (3) and since

$$\lim_{t\to+\infty} tM(t^{\frac{2-N}{2}}) = +\infty,$$

by continuity we have that there exists $\bar{t} > 0$ such that $\bar{t}^2 M(\bar{t}^{2-N}\int_{\mathbb{R}^N} |\nabla v|^2) = 1$. The function $u : \mathbb{R}^N \to \mathbb{R}$ defined as follows:

$$x \in \mathbb{R}^N \to v(\bar{t}x) \in \mathbb{R},$$

satisfies the equalities

$$\begin{cases} M(\int_{\mathbb{R}^N}|\nabla u|^2) = \frac{1}{\bar{t}^2} \\ -\Delta u(x) = -\bar{t}^2\Delta v(\bar{t}x) = \bar{t}^2 g(v(\bar{t}x)) = \bar{t}^2 g(u(x)), \end{cases}$$

and then it is a solution of $(\mathcal{K})$. □

Remark 1.1. *We point out that, since the only moment in which we use hypothesis* (3) *is to determine the rescaling parameter* $\bar{t}$*, we can relax our assumption just requiring that*

$$\inf_{t\geqslant 0} tM\left(t^{\frac{2-N}{2}}\int_{\mathbb{R}^N}|\nabla v|^2\right) \leqslant 1. \tag{6}$$

Proof of Theorem 0.5 As in the proof of Theorem 0.4 we look for a solution to the equation

$$t^2\left(a + bf(t^{2-N}\int_{\mathbb{R}^N}|\nabla v|^2)\right) = 1.$$

Taking into account the previous remark, it is enough to prove that

$$\inf_{t\geqslant 0}\Psi(t) \leqslant 1,$$

where $\Psi(t) := t\left(a + bf(t^{\frac{2-N}{2}}\int_{\mathbb{R}^N}|\nabla v|^2)\right)$. Set

$$\bar{h} = f\left((2a)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}|\nabla v|^2\right)$$

and $\delta_1 = \frac{a}{\bar{h}}$. It is easy to verify that, if $b \leqslant \delta_1$, then $\Psi(1/2a) \leqslant 1$.
Now suppose that (4) holds. We deduce that

$$\liminf_{t\to+\infty} tf\left(t^{\frac{2-N}{2}}\int_{\mathbb{R}^N}|\nabla v|^2\right) = 0.$$

Let $\bar{t}$ such that $\bar{t}f\left(\bar{t}^{\frac{2-N}{2}}\int_{\mathbb{R}^N}|\nabla v|^2\right) \leqslant \frac{1}{2b}$ and choose $a \leqslant \delta_2 = \frac{1}{2\bar{t}}$. Again we have that $\Psi(\bar{t}) \leqslant 1$. □

2 Ground state solution

In this section we use a variational approach which requires some preliminaries. In next subsection we will use the same arguments as in [2] to modify the nonlinearity g in such a way we can study equation $(\mathcal{K})$ looking for critical points of a suitable functional.

2.1 Functional framework

Define $s_0 := \min\{s \in [\zeta, +\infty[\mid g(s) = 0\}$ ($s_0 = +\infty$ if $g(s) \neq 0$ for any $s \geqslant \zeta$). We set $\tilde{g} : \mathbb{R} \to \mathbb{R}$ the function such that

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0]; \\ 0 & \text{on } \mathbb{R}_+ \setminus [0, s_0]; \\ (g(-s) - ms)^+ - g(-s) & \text{on } \mathbb{R}_-. \end{cases} \tag{7}$$

By the strong maximum principle, if u is a nontrivial solution of $(\mathcal{K})$ with $\tilde{g}$ in the place of g, then $0 < u < s_0$ and so it is a positive solution of $(\mathcal{K})$. Therefore we can suppose that g is defined as in (7), so that **(g1)**, **(g2)**, **(g4)** and the following limit

$$\lim_{s\to\pm\infty} \frac{g(s)}{s^{2^*-1}} = 0 \tag{8}$$

hold.
We set

$$g_1(s) := \begin{cases} (g(s)+ms)^+, & \text{if } s \geqslant 0,\\ 0, & \text{if } s < 0, \end{cases}$$
$$g_2(s) := g_1(s) - g(s), \quad \text{for } s \in \mathbb{R}.$$

Since

$$\lim_{s\to 0} \frac{g_1(s)}{s} = 0,$$
$$\lim_{s\to\pm\infty} \frac{g_1(s)}{s^{2^*-1}} = 0, \tag{9}$$

and

$$g_2(s) \geqslant ms, \quad \forall s \geqslant 0, \tag{10}$$

by some computations, we have that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$g_1(s) \leqslant C_\varepsilon s^{2^*-1} + \varepsilon g_2(s), \quad \forall s \geqslant 0. \tag{11}$$

If we set

$$G_i(t) := \int_0^t g_i(s)\, ds, \quad i = 1, 2,$$

then, by (10) and (11), we have

$$G_2(s) \geqslant \frac{m}{2} s^2, \quad \forall s \in \mathbb{R} \tag{12}$$

and for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$G_1(s) \leqslant \frac{C_\varepsilon}{2^*} s^{2^*} + \varepsilon G_2(s), \quad \forall s \in \mathbb{R}. \tag{13}$$

We define the functional

$$I(u) := \frac{1}{2}\tilde{M}(\|u\|^2) - \int_{\mathbb{R}^3} G(u)$$

where we are denoting by $\|\cdot\|$ the norm $\left(\int_{\mathbb{R}^3} |\nabla\cdot|^2\right)^{\frac{1}{2}}$ of the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, which is the closure of the compactly supported smooth functions with respect to the norm $\|\cdot\|$. The previous functional is C^1 in $H^1(\mathbb{R}^N)$, being $H^1(\mathbb{R}^N)$ the closure of the compactly supported smooth functions with respect to the norm

$$\|\cdot\|^2_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^3} |\nabla\cdot|^2 + \int_{\mathbb{R}^3} |\cdot|^2.$$

We will look for critical points of the functional I inside

$$H^1_r(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \mid u \text{ is radial}\},$$

which is a natural constraint for the functional I by Palais' principle of symmetric criticality. By standard variational arguments, it is easy to prove that any critical point of I corresponds to a weak solution of the equation. By the maximum principle we will get a positive solution.

2.2 Existence of a ground state solution

We look for a ground state solution to

$$-(a + b\|u\|^2)\Delta u = g(u), \quad u : \mathbb{R}^N \to \mathbb{R},\ N = 3,\, 4.$$

A ground state of (5) is a nontrivial solution $\bar{u} \in H^1(\mathbb{R}^N)$ such that, if $v \in H^1(\mathbb{R}^N)$ is another nontrivial solution of (5), then

$$I(\bar{u}) \leqslant I(v),$$

where $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ is the functional of the action related with (5), namely

$$I(u) = \frac{1}{2}\left(a + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u|^2\right)\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u).$$

Usually a standard technique to find a ground state consists in looking for minimizers of the functional of the action restricted to a natural constraint which contains all the possible solutions. A candidate to play this role is the following Pohozaev set

$$\mathcal{P} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid P(u) = 0\}$$

where for any $u \in H^1(\mathbb{R}^N)$

$$P(u) = a\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u|^2 + b\frac{N-2}{2N}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} G(u).$$

Actually the equality $P(u) = 0$ is nothing but the Pohozaev identity related with equation.

We will prove Theorem 0.6 following this scheme:

step 1: we show that $\mathcal{P}$ is a C^1 manifold containing all the possible solutions of equation (5);

step 2: we prove that $\mathcal{P}$ is a natural constraint, in the sense that every critical point of I restricted to $\mathcal{P}$ is a critical point of I;

step 3: we show that $I|_{\mathcal{P}}$ is bounded below and

$$\mu = \inf_{u \in \mathcal{P}} I(u) = \inf_{u \in \mathcal{P}} \frac{1}{N}\left(a\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{(4-N)b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2\right)$$

is achieved.

It is easy to see that P is a C^1 functional. Moreover $\mathcal{P}$ is nondegenerate in the following sense:

$$\forall u \in \mathcal{P} : P'(u) \neq 0$$

so that $\mathcal{P}$ is a C^1 manifold of codimension one. Indeed, suppose by contradiction that $u \in \mathcal{P}$ and $P'(u) = 0$, namely u is a solution of the equation

$$-\left(a\frac{N-2}{N} + 2b\frac{N-2}{N}\|u\|^2\right)\Delta u = g(u). \tag{14}$$

As a consequence, u satisfies the Pohozaev identity referred to (14), that is

$$a\frac{(N-2)^2}{2N^2}\int_{\mathbb{R}^N} |\nabla u|^2 + b\frac{(N-2)^2}{N^2}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 = \int_{\mathbb{R}^N} G(u). \tag{15}$$

Since $P(u) = 0$, by (15) we get

$$-2a\int_{\mathbb{R}^N} |\nabla u|^2 + b(N-4)\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 = 0$$

and we conclude that $u = 0$: absurd since $u \in \mathcal{P}$. So $\mathcal{P}$ is a C^1 manifold. It obviously contains all the solutions to (5) since every solution satisfies the Pohozaev identity $P(u) = 0$.

Now we pass to prove that $\mathcal{P}$ is a natural constraint for I. Suppose that $u \in \mathcal{P}$ is a critical point of the functional $I|_{\mathcal{P}}$. Then, there exists $\lambda \in \mathbb{R}$ such that

$$I'(u) = \lambda P'(u),$$

that is

$$-(a + b\|u\|^2)\Delta u - g(u) = -\lambda(a\frac{N-2}{N} + 2b\frac{N-2}{N}\|u\|^2)\Delta u - \lambda g(u).$$

As a consequence, u satisfies the following Pohozaev identity

$$P(u) = \lambda a\frac{(N-2)^2}{2N^2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b\frac{(N-2)^2}{N^2}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \lambda\int_{\mathbb{R}^N} G(u)$$

which, since $P(u) = 0$, can be written

$$\lambda\left(-2a\int_{\mathbb{R}^N} |\nabla u|^2 + b(N-4)\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2\right) = 0.$$

Since $u \neq 0$, we deduce that $\lambda = 0$, and we conclude.

Now it remains to show that μ is achieved.

By the well known properties of the Schwarz symmetrization, we are allowed to work on the functional space $H^1_r(\mathbb{R}^N)$ as showed by the following

Lemma 2.1. *For any $u \in \mathcal{P}$ there exists $\tilde{u} \in \mathcal{P} \cap H^1_r(\mathbb{R}^N)$ such that $I(\tilde{u}) \leqslant I(u)$*

Proof Let $u \in \mathcal{P}$ and set $u^* \in H^1_r(\mathbb{R}^N)$ its symmetrized. It is easy to see that there exists $0 < \tilde\theta \leqslant 1$ such that $\tilde u := u^*(\cdot/\tilde\theta) \in \mathcal{P} \cap H^1_r(\mathbb{R}^N)$ and

$$\begin{aligned} I(\tilde u) &= \frac{1}{N}\left(a\int_{\mathbb{R}^N}|\nabla \tilde u|^2 + \frac{(4-N)b}{4}\left(\int_{\mathbb{R}^N}|\nabla \tilde u|^2\right)^2\right) \\ &= a\frac{\tilde\theta^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla u^*|^2 + b\frac{(4-N)\tilde\theta^{2(N-2)}}{4N}\left(\int_{\mathbb{R}^N}|\nabla u^*|^2\right)^2 \\ &\leqslant \frac{a}{N}\int_{\mathbb{R}^N}|\nabla u^*|^2 + \frac{(4-N)b}{4N}\left(\int_{\mathbb{R}^N}|\nabla u^*|^2\right)^2 \\ &\leqslant I(u). \end{aligned}$$

□

Before we proceed with the proof of the main result, another preliminary result is required

Lemma 2.2. $\mu := \inf\{I(v) \mid v \in \mathcal{P}\} > 0.$

Proof If $u \in \mathcal{P}$, then, by (13), we have

$$\begin{aligned} C\|u\|^2 &\leqslant a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{N-2}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^2 + N(1-\varepsilon)\int_{\mathbb{R}^N} G_2(u) \\ &\leqslant NC_\varepsilon\int_{\mathbb{R}^N}|u|^{2^*} \leqslant C'\|u\|^{2^*} \end{aligned}$$

where $\varepsilon < 1, C_\varepsilon, C$ and C' are suitable positive constants. We deduce that there exists a positive constant C'' such that $\|u\| \geqslant C''$ for any $u \in \mathcal{P}$. The conclusion then follows once one observes that $I|_{\mathcal{P}}(u) \geqslant \tilde C\|u\|^2$. □

Now let $(u_n)_n$ be a minimizing sequence for $I|_{\mathcal{P}}$ in $H^1_r(\mathbb{R}^N)$, namely

$$\{u_n\}_n \subset \mathcal{P} \cap H^1_r(\mathbb{R}^N), \quad I(u_n) \to \mu. \tag{16}$$

Obviously $\|u_n\|$ is bounded. Moreover, since $\{u_n\}_n \subset \mathcal{P}$, certainly, by (13), there exist $0 < \varepsilon < 1$ and $C_\varepsilon > 0$ such that

$$a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{N-2}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^2 + N(1-\varepsilon)\int_{\mathbb{R}^N} G_2(u) \leqslant C_\varepsilon N\|u_n\|_{2^*}^{2^*},$$

and then we deduce also the boundedness of the L^2-norm of $\{u_n\}_n$ by the continuous Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and (12).

Let $u \in H^1_r(\mathbb{R}^N)$ be the function such that, up to subsequences,

$$u_n \rightharpoonup u, \text{ weakly in } H^1(\mathbb{R}^N). \tag{17}$$

We are going to prove that there exists $\bar\theta > 0$ such that

$$\bar u \in \mathcal{P} \text{ and } I(\bar u) = \mu$$

where $\bar u := u(\cdot/\bar\theta)$.
Actually, by compactness due to the radial symmetry, from the weak convergence (17) we deduce

$$\lim_n \int_{\mathbb{R}^N} G_1(u_n) = \int_{\mathbb{R}^N} G_1(u). \tag{18}$$

Of course, $u \neq 0$. Otherwise, by (18) and since $u_n \in \mathcal{P}$ for any $n \geqslant 1$, we should have that

$$0 \leqslant \limsup_n a\frac{N-2}{2}\|u_n\|^2 \leqslant N \lim_n \int_{\mathbb{R}^N} G_1(u_n) = 0,$$

which, by (16), contradicts Lemma 2.2.
By (18), the lower semicontinuity of the $\mathcal{D}^{1,2}(\mathbb{R}^N)-$norm and the Fatou lemma, we have

$$\begin{aligned}
& a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{N-2}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^2 + N\int_{\mathbb{R}^N}G_2(u) \\
& \leqslant \liminf_n \left(a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 + b\frac{N-2}{2}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\right)^2 + N\int_{\mathbb{R}^N}G_2(u_n)\right) \\
& = \lim_n N\int_{\mathbb{R}^N}G_1(u_n) = N\int_{\mathbb{R}^N}G_1(u).
\end{aligned}$$

Let $0 < \bar{\theta} \leqslant 1$ such that $\bar{u} = u(\cdot/\bar{\theta}) \in \mathcal{P}$. Using the lower semicontinuity of the $\mathcal{D}^{1,2}(\mathbb{R}^N)-$norm, we infer that

$$\begin{aligned}
I(\bar{u}) &= a\frac{\bar{\theta}^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{(4-N)\bar{\theta}^{2(N-2)}}{4N}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^2 \\
& \leqslant \liminf_n \frac{a}{N}\int_{\mathbb{R}^N}|\nabla u_n|^2 + \frac{b(4-N)}{4N}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\right)^2 = \lim_n I(u_n) = \mu
\end{aligned}$$

and then we conclude.

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